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HOMOGENIZATION OF A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM MODELING GALVANIC CURRENTS*

Y. S. BHAT† AND S. MOSKOW†

Abstract. We study a nonlinear elliptic boundary value problem arising from electrochemistry in the study of heterogeneous electrode surfaces. The boundary condition is of exponential type (Butler–Volmer) and has a periodic structure. We find a limiting or effective problem as the period approaches zero, along with a first order correction. We establish convergence estimates and provide numerical experiments.

Key words. galvanic corrosion, homogenization, nonlinear elliptic boundary value problem, Butler–Volmer boundary condition

AMS subject classifications. 35J65, 35Q72

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1. Introduction.

In the electrochemistry community there is much interest in the study of galvanic interactions on heterogeneous surfaces [9], [10]. When two different metals in electrical contact, referred to as anode and cathode, are immersed in an electrolytic solution, the difference in rest potential generates an electron flow. This electron flow is called a galvanic current and may lead to a deterioration (corrosion) of the anode.

In Figure 1.1 a strip of silver (Ag) and a strip of zinc (Zn) have been immersed in a saltwater solution. The zinc strip gives up electrons to the silver strip. The silver strip is said to be *cathodic*, and *reduction* takes place (Ag gains electrons). Simultaneously *oxidation* takes place at the zinc strip; zinc loses electrons and is said to be *anodic*. Zinc dissolves into the solution, the zinc electrode is being corroded, and the electron flow is known as galvanic current. The driving force of the electron transport process is the difference in potential of the two metals involved. See [12] for a complete introduction to the subject.

Here we study the electrostatic problem on a surface where anodes are arranged periodically in a cathodic matrix. Mathematically the potential is modeled as a function, ϕ, over a Euclidean domain Ω. Part of the boundary of Ω is electrochemically active while the rest of the boundary is inert. It is the active region of the boundary that is made up of anodic and cathodic portions. The potential over both of these regions satisfies an exponential boundary condition of Butler and Volmer but with different material parameters on each portion. In [9] the authors study such a problem numerically, using finite elements. Additionally various interesting aspects of the two dimensional, homogeneous model with the Butler–Volmer condition have been analyzed in [3], [6], and [15]. To the best of our knowledge, however, studies coming from the applied mathematics community have been restricted to two dimensions. The main reason for this is that one can bound exponentials of the two dimensional weak solution on the boundary by using an Orlicz estimate [14], [15]. Such an estimate would require more than H^1 regularity in higher dimensions. In this paper, we attempt

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†Department of Mathematics, University of Florida, Gainesville, FL 32611 (ybhat@math.ufl.edu, moskow@math.ufl.edu).

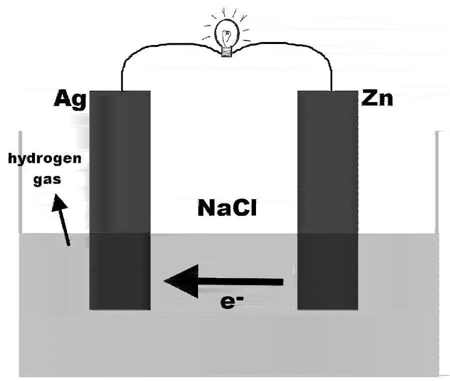

FIG. 1.1. *Zinc loses electrons to silver.*

to treat a periodically heterogeneous problem, in two and three dimensions, from the point of view of homogenization theory.

The three dimensional model is as follows. The domain Ω is of cylindrical shape with its base being some two dimensional domain. The bottom base is assumed to contain a periodic arrangement of islands (anodes). We call this collection of islands $\partial\Omega_A$ and the remainder of the bottom of the base $\partial\Omega_C$ (cathodic plane). The electrolytic voltage potential, ϕ, satisfies the nonlinear elliptic boundary value problem

$$\begin{aligned} &\Delta\phi = 0 \text{ in } \Omega, \\ &-\frac{\partial\phi}{\partial n} = J_A[e^{\alpha_{aa}(\phi - V_A)} - e^{-\alpha_{ac}(\phi - V_A)}] \text{ on } \partial\Omega_A, \\ &-\frac{\partial\phi}{\partial n} = J_C[e^{\alpha_{ca}(\phi - V_C)} - e^{-\alpha_{cc}(\phi - V_C)}] \text{ on } \partial\Omega_C, \\ &-\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega \setminus \{\partial\Omega_A \cup \partial\Omega_C\}, \end{aligned} \tag{1.1}$$

where $\alpha_{aa}, \alpha_{ac}, \alpha_{ca}, \alpha_{cc}$ are the transfer coefficients and it is assumed that the sums $(\alpha_{aa} + \alpha_{ac})$ and $(\alpha_{ca} + \alpha_{cc})$ are equal to one. The positive constants J_A, J_C are the anodic and cathodic polarization parameters and V_A, V_C are the anodic and cathodic rest potentials, respectively. Note that $\nabla\phi$ represents galvanic current. These boundary conditions are the so-called Butler–Volmer exponential boundary conditions.

In the numerical studies of [9], the authors observed that for fixed ratios of anodic to cathodic areas on the bottom base, the resulting current increased approximately linearly with the length of the perimeter between the two regions, and they hypothesized that it is the ratio of anodic area to perimeter that determines the size of the resulting current.

As a special case of increasing perimeter with approximately fixed area fraction, we consider a periodic structure with period approaching zero. Our goal is to expand the solution asymptotically with respect to the period size. Convergence results involving these approximations could provide insight into the behavior of the current for small period size and possibly lead to techniques for computing approximate solutions to (1.1).

We model the periodic structure by letting

$$f(y, v) = \lambda(y)[e^{\alpha(y)(v - V(y))} - e^{-(1-\alpha(y))(v - V(y))}]$$

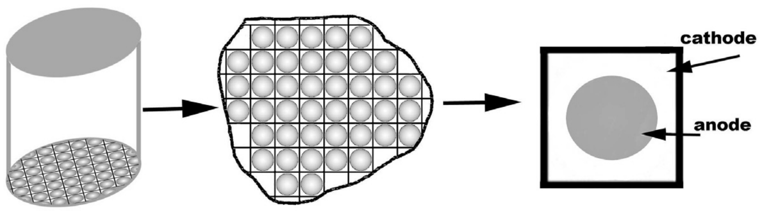

FIG. 1.2. *The base is a heterogeneous surface.*

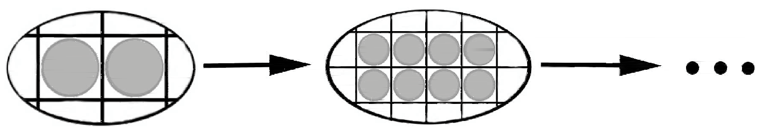

FIG. 1.3. *Perimeter increases while anodic area fraction stays constant.*

for any $v \in R$ and $y \in Y$, the boundary period cell, which for simplicity we take to be the unit square; $Y = [0,1] \times [0,1]$. Here λ, α, and V are all piecewise smooth Y-periodic functions. We also assume there exist constants $\lambda_0, \Lambda_0, \alpha_0, A_0$, and V_0 such that

$$0 \;<\; \lambda_0 \;\leq\; \lambda(y) \;\leq\; \Lambda_0, \tag{1.2}$$

$$0 < \alpha_0 \leq \alpha(y) \leq A_0 < 1, \tag{1.3}$$

and

$$|V(y)| \leq V_0. \tag{1.4}$$

See [3] and [15] for an analysis of when $\lambda < 0$.

Consider the problem

$$\begin{aligned} \Delta u_\epsilon &= 0 \text{ in } \Omega, \\ -\frac{\partial u_\epsilon}{\partial n} &= f\left(\frac{x}{\epsilon}, u_\epsilon\right) \text{ on } \Gamma, \\ -\frac{\partial u_\epsilon}{\partial n} &= 0 \text{ on } \partial\Omega \setminus \Gamma. \end{aligned} \tag{1.5}$$

As is typical in homogenization problems, one expects that as $\epsilon \to 0$, the solutions will converge in some sense to a solution of a problem with an averaged boundary condition. Define $f_0(v)$ to be a cell average of $f(y,v)$, that is,

$$f_0(v) = \int_Y f(y,v)dy.$$

Consider the candidate for the homogenized problem

$$\begin{aligned} \Delta u_0 &= 0 \text{ in } \Omega, \\ -\frac{\partial u_0}{\partial n} &= f_0(u_0) \text{ on } \Gamma, \\ -\frac{\partial u_0}{\partial n} &= 0 \text{ on } \partial\Omega \setminus \Gamma. \end{aligned} \tag{1.6}$$

Remark. If, as is the case in [9], $Y = Y_1 \bigcup Y_2$ and the functions λ, α, V are piecewise constant, each taking on the values λ_i, α_i, V_i, respectively, in Y_i, then

$$f_0(v) = |Y_1|\lambda_1 \left[e^{\alpha_1(v-V_1)} - e^{-(1-\alpha_1)(v-V_1)}\right] + |Y_2|\lambda_2 \left[e^{\alpha_2(v-V_2)} - e^{-(1-\alpha_2)(v-V_2)}\right].$$

That is, the above homogenized boundary condition would depend on the volume fraction of anodic to cathodic regions.

The paper is organized as follows. In section 2, we show the existence and uniqueness of weak solutions to (1.5) and (1.6) in any dimension and discuss regularity. In section 3, we introduce a correction term. This correction term satisfies a heterogeneous boundary condition but is *linear*. For dimension $n = 2$, we prove convergence estimates for our approximation. For $n = 3$, we show a partial result; the same convergence estimates hold if one has a priori knowledge that the solutions to (1.5) are continuous and uniformly bounded. In section 4, we test the accuracy of our approximation with numerical experiments.

2. Existence and uniqueness. In this section we show that the energy minimization forms of the nonlinear problem (1.5) and (1.6) have unique solutions in $H^1(\Omega)$ in any dimension. Some elements of the proof are similar to those in [6] and [15]. For a given ϵ, define the energy functional

$$E_\epsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \ \ dx + \int_\Gamma F\left(\frac{x}{\epsilon}, v\right) d\sigma_x, \tag{2.1}$$

where

$$F(y, v) = \frac{\lambda(y)}{\alpha(y)} e^{\alpha(y)(v-V(y))} + \frac{\lambda(y)}{1-\alpha(y)} e^{-(1-\alpha(y))(v-V(y))}.$$

We show the existence and uniqueness of a minimizer of (2.1). Formally, we show the existence of a function $u_\epsilon \in H^1(\Omega)$ such that

$$E_\epsilon(u_\epsilon) = \min_{u\in H^1(\Omega)} E_\epsilon(u). \tag{2.2}$$

Note that E_ϵ is not necessarily bounded on all of $H^1(\Omega)$ (unless $n = 2$ for which we can use an Orlicz estimate). However, this does not pose a problem. We set E_ϵ equal to (2.1), where it is well defined, and to $+\infty$, where it is not, as in [4, p. 444]. In the two dimensional case of the model, due to the boundedness of E_ϵ on $H^1(\Omega)$, direct calculation shows u_ϵ satisfies the variational form of (1.5),

$$\int_\Omega \nabla u_\epsilon \cdot \nabla v \ dx = -\int_\Gamma f(x/\epsilon, u_\epsilon) v \ d\sigma_x \quad \text{ for any } v \in H^1(\Omega). \tag{2.3}$$

In the three dimensional case, if u_ϵ is an energy minimizer, we will have that

$$\int_\Gamma F(x/\epsilon, u_\epsilon) \ d\sigma_x < \infty, \tag{2.4}$$

and hence by the positivity of each term of $F(x/\epsilon, u_\epsilon)$, we have that each term is separately in $L^1(\Omega)$. Therefore,

$$E_\epsilon(u_\epsilon + tv) < \infty \tag{2.5}$$

for any $t \in R$ and for any v which is smooth on Γ. Standard arguments then show that u_ϵ satisfies

$$\int_\Omega \nabla u_\epsilon \cdot \nabla v \, dx = -\int_\Gamma f(x/\epsilon, u_\epsilon) v \, d\sigma_x \quad \text{for any } v \in C^\infty(\bar{\Omega}).$$

Additionally, if we know that $u_\epsilon \in C^0(\bar{\Omega})$, then $f(x/\epsilon, u_\epsilon)$ is bounded and hence clearly in $H^{-1/2}(\Gamma)$. So by the density of $C^\infty(\bar{\Omega})$ functions in $H^1(\Omega)$, u_ϵ in this case would satisfy

$$\int_\Omega \nabla u_\epsilon \cdot \nabla v \, dx = -\int_\Gamma f(x/\epsilon, u_\epsilon) v \, d\sigma_x \quad \text{for any } v \in H^1(\Omega). \tag{2.6}$$

Consider also the functional

$$E_0(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx + \int_\Gamma F_0(v) \, d\sigma_x, \tag{2.7}$$

where

$$F_0(v) = \int_Y F(y, v) dy.$$

Here again the energy E_0 is not necessarily bounded, but as before we set E_0 equal to (2.7), where it is well defined, and to $+\infty$, where it is not. Direct calculations show that a minimizer u_0 of (2.7) will satisfy

$$\int_\Omega \nabla u_0 \cdot \nabla v \, dx = -\int_\Gamma f(u_0) v \, d\sigma_x \quad \text{for any } v \in H^1(\Omega), \tag{2.8}$$

assuming u_0 is continuous (actually we will see that u_0 is a constant).

Theorem 2.1 (existence and uniqueness of the minimizer). *Let E_ϵ be defined by* (2.1), *where λ, α, and V satisfy* (1.2)–(1.4). *Then there exists a unique function $u_\epsilon \in H^1(\Omega)$ satisfying*

$$E_\epsilon(u_\epsilon) = \min_{u \in H^1(\Omega)} E_\epsilon(u).$$

Proof. Note that

$$\frac{\partial^2}{\partial v^2} F(y, v) = \lambda(y)\alpha(y) e^{\alpha(y)(v - V(y))} + \lambda(y)(1 - \alpha(y)) e^{-(1-\alpha(y))(v - V(y))}.$$

Since $\lambda > 0, \alpha > 0$, and $1 - \alpha > 0$ we have that $\frac{\partial^2}{\partial v^2} F > 0$. Clearly the partial derivative is bounded below. That is, there exists a constant c_0, independent of y and v, such that

$$\frac{\partial^2}{\partial v^2} F(y, v) \geq c_0 > 0.$$

Since F is smooth in the second variable, for any $v, w \in H^1(\Omega)$ and for any y, there exists some ξ between $v + w$ and $v - w$ such that

$$F(y, v + w) + F(y, v - w) - 2F(y, v) = \frac{\partial^2}{\partial v^2} F(y, \xi) w^2,$$

which from the lower bound yields

$$F\left(\frac{x}{\epsilon}, v+w\right) + F\left(\frac{x}{\epsilon}, v-w\right) - 2F\left(\frac{x}{\epsilon}, v\right) \geq c_0 w^2;$$

whence

$$\begin{aligned} E_\epsilon(v+w) + E_\epsilon(v-w) - 2E_\epsilon(v) &\geq \int_\Omega |\nabla w|^2 \, dx + c_0 \int_\Gamma w^2 d\sigma_x \\ &\geq \tilde{c}_0 \|w\|^2_{H^1(\Omega)}, \end{aligned} \tag{2.9}$$

where the last inequality follows by a variant of Poincairé. Now let $\{u_\epsilon^n\}_{n=1}^\infty$ be a minimizing sequence, that is,

$$E_\epsilon(u_\epsilon^n) \to \inf_{u \in H^1(\Omega)} E_\epsilon(u) \qquad \text{as} \qquad n \to \infty.$$

Since all the terms of (2.1) are nonnegative, clearly

$$\inf_{u \in H^1(\Omega)} E_\epsilon(u) > -\infty.$$

Note that without loss of generality we can choose the minimizing sequence so that all terms have finite energy (since $\inf_{u \in H^1(\Omega)} E_\epsilon(u) \leq E(0)$ and $E(0)$ is bounded independently of ϵ). Let

$$v = \frac{u_\epsilon^n + u_\epsilon^m}{2}$$

and

$$w = \frac{u_\epsilon^n - u_\epsilon^m}{2}.$$

Then $v + w = u_\epsilon^n$ and $v - w = u_\epsilon^m$, so (2.9) implies

$$E_\epsilon(v+w) + E_\epsilon(v-w) - 2E_\epsilon(v) \geq \frac{\tilde{c}_0}{4} \|u_\epsilon^n - u_\epsilon^m\|^2_{H^1(\Omega)},$$

which implies

$$E_\epsilon(u_\epsilon^n) + E_\epsilon(u_\epsilon^m) - 2 \inf_{v \in H^1(\Omega)} E_\epsilon(v) \geq \frac{\tilde{c}_0}{4} \|u_\epsilon^n - u_\epsilon^m\|^2_{H^1(\Omega)}.$$

Now if we let $m, n \to \infty$, we see that $\{u_\epsilon^n\}_n$ is a Cauchy sequence in the Hilbert space $H^1(\Omega)$. Define u_ϵ to be its limit in $H^1(\Omega)$. Then we have

$$u_\epsilon^n \to u_\epsilon \ \text{ in } \ H^1(\Omega),$$

which by the trace theorem implies

$$u_\epsilon^n \to u_\epsilon \ \text{ in } \ L^2(\Gamma),$$

which implies (see [13, p. 68]) there exists a subsequence $\{u_\epsilon^{n_k}\}_k$, which we label $\{u_\epsilon^k\}_k$, such that

$$u_\epsilon^k \to u_\epsilon \ \text{ a.e. in } \ \Gamma.$$

Since F is smooth in the second variable and $u_\epsilon^k \to u_\epsilon$ a.e. in Γ we have that

$$F\left(\frac{x}{\epsilon}, u_\epsilon\right) = \lim_{k\to\infty} F\left(\frac{x}{\epsilon}, u_\epsilon^k\right) \text{ a.e.}$$

Now note that clearly $F(\frac{x}{\epsilon}, u_\epsilon^k) > 0$ for any k. So, by Fatou's lemma we have

$$\int_\Gamma F\left(\frac{x}{\epsilon}, u_\epsilon\right) d\sigma_x \leq \liminf_{k\to\infty} \int_\Gamma F\left(\frac{x}{\epsilon}, u_\epsilon^k\right) d\sigma_x.$$

Thus from this and the fact that $u_\epsilon^k \to u_\epsilon$ in $H^1(\Omega)$, we can conclude that

$$\begin{aligned} E_\epsilon(u_\epsilon) &\leq \liminf_{k\to\infty} E_\epsilon(u_\epsilon^k) \\ &= \lim_{k\to\infty} E_\epsilon(u_\epsilon^k) \\ &= \inf_{u\in H^1(\Omega)} E_\epsilon(u). \end{aligned}$$

Hence,

$$E_\epsilon(u_\epsilon) = \inf_{u\in H^1(\Omega)} E_\epsilon(u).$$

So we have shown the existence of a minimizer.

Suppose u_ϵ and w_ϵ are both minimizers of the energy functional, i.e.,

$$E_\epsilon(u_\epsilon) = \inf_{u\in H^1(\Omega)} E_\epsilon(u) = E_\epsilon(w_\epsilon).$$

Now if we let

$$v = (u_\epsilon + w_\epsilon)/2$$

and

$$w = (u_\epsilon - w_\epsilon)/2,$$

then substituting v and w into (2.9) yields

$$E_\epsilon(u_\epsilon) + E_\epsilon(w_\epsilon) - 2E_\epsilon\left(\frac{u_\epsilon + w_\epsilon}{2}\right) \geq \frac{\tilde{c}_0}{4}\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)}.$$

However, this implies

$$\frac{\tilde{c}_0}{4}\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)} \leq E_\epsilon(u_\epsilon) + E_\epsilon(w_\epsilon) - 2\inf_{u\in H^1(\Omega)} E_\epsilon(u) = 0.$$

Hence $u_\epsilon = w_\epsilon$ in $H^1(\Omega)$. Thus we have shown the uniqueness of the minimizer. □

Note that this argument can be generalized to address the n-dimensional problem, i.e., the case in which we have $\Omega \subset R^n, \Gamma \subset R^{n-1}$ with boundary period cell $Y = [0,1]^{n-1}$. The existence and uniqueness of a minimizer u_0 of E_0 follows from the same proof.

COROLLARY 2.2. *There exists a constant C, depending on $\Lambda_0, a_0, A_0,$ and V_0 but independent of ϵ, such that*

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C,$$

where u_ϵ *is a weak solution to* (1.5).

Proof. Consider the function $v \equiv 0$. Then

$$E_\epsilon(v) = E_\epsilon(0) = \int_\Gamma F\left(\frac{x}{\epsilon}, 0\right) d\sigma_x \leq M$$

for M independent of ϵ (but depending on Λ_0, a_0, A_0, and V_0). Then since u_ϵ is a minimizer,

$$E_\epsilon(u_\epsilon) \leq E_\epsilon(0) \leq M.$$

Since both terms in E_ϵ are positive,

$$\|\nabla u_\epsilon\|^2_{L^2(\Omega)} \leq M.$$

We also have that

$$\int_\Gamma F\left(\frac{x}{\epsilon}, u_\epsilon\right) d\sigma_x \leq M.$$

By examining the form of $F(y, v)$, we see that there exists some constant d, depending on Λ_0, a_0, and A_0 but independent of ϵ and x, such that

$$d\left|u_\epsilon - V\left(\frac{x}{\epsilon}\right)\right| \leq F\left(\frac{x}{\epsilon}, u_\epsilon\right).$$

Hence,

$$\int_\Gamma \left|u_\epsilon - V\left(\frac{x}{\epsilon}\right)\right| d\sigma_x \leq \frac{M}{d},$$

which by the boundedness of V implies that

$$\int_\Gamma |u_\epsilon| d\sigma_x \leq \tilde{M},$$

where $\tilde{M}$ is independent of ϵ. One variant of the Poincaré inequality says that there exists $\hat{C}$ such that

$$\left\| u_\epsilon - \int_\Gamma u_\epsilon d\sigma_x \right\|_{L^2(\Omega)} \leq \hat{C}\|\nabla u_\epsilon\|_{L^2(\Omega)}.$$

Finally, the reverse triangle inequality yields

$$\|u_\epsilon\|_{L^2(\Omega)} \leq \hat{C}\|\nabla u_\epsilon\|_{L^2(\Omega)} + \tilde{M},$$

which proves the corollary. ∎

We conclude this section with a short discussion of the regularity of the solutions u_ϵ and u_0. For the two dimensional case of this problem, i.e., when the medium is layered as in [6], [15] (see Figure 2.1), using embeddings of Sobolev spaces into Orlicz spaces we can show that $f(x_2/\epsilon, u_\epsilon)$ and $f_0(u_0)$ are bounded in $L^2(\Gamma)$ independently of ϵ. The Orlicz estimate used for this two dimensional result is the following (see [14], [15]): There exists a constant C such that for any $v \in H^1(\Omega)$ and any real β we have

$$\int_\Gamma e^{\beta|v|} dx_2 \leq e^{C\beta^2(\|v\|_{H^1(\Omega)}+1)}(|\Gamma| + 1).$$

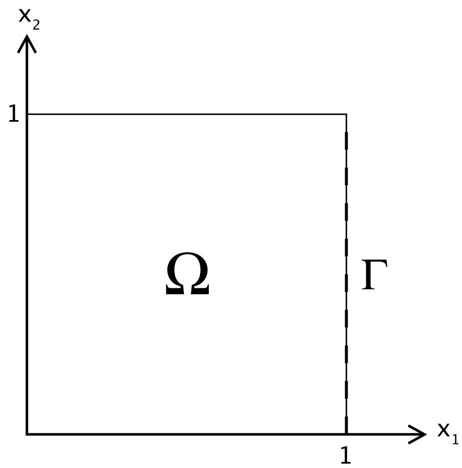

FIG. 2.1. *Two dimensional analogue.*

Then from standard elliptic regularity theory this implies that u_ϵ and u_0 are in $H^{3/2}(\Omega)$, with the norm bounded independently of ϵ. By the trace theorem we then obtain bounds for u_ϵ and u_0 in $H^1(\Gamma)$. Since Γ is one dimensional it follows that u_ϵ and u_0 are continuous on Γ and bounded pointwise, and their tangential derivatives are bounded in $L^2(\Gamma)$. For the homogenized solution we have much more regularity; u_0 is in fact the constant that satisfies $f_0(u_0) = 0$. For nonzero boundary conditions on the inactive region, u_0 would still be a smooth bounded function. So for the two dimensional version of this problem we have the following lemma.

LEMMA 2.3. *If $\Omega \subset R^2$ is a rectangle and Γ is an edge, then $u_\epsilon \in C(\bar{\Omega})$, where u_ϵ is a weak solution of* (1.5). *Furthermore, there exists a constant D, the value of which does not depend on ϵ, such that*

$$\|u_\epsilon(x)\|_{C(\bar{\Omega})} \le D.$$

3. Convergence estimates and corrections. To show u_ϵ converges to u_0 we will add a correction term and prove estimates in terms of powers of ϵ. The convergence of u_ϵ to u_0 when $n = 2$ will then easily follow from this. We will see that the convergence is strong in $H^1(\Omega)$ and of the order of $\sqrt{\epsilon}$. The same estimate holds when $n = 3$ if we know that the solutions are continuous and uniformly bounded.

Let u_0 be a minimizer of (2.7) and define the correction $u_\epsilon^{(1)}$ to satisfy

$$\begin{aligned} &\Delta u_\epsilon^{(1)} = 0 \text{ in } \Omega, \\ &-\frac{\partial u_\epsilon^{(1)}}{\partial n} = \frac{1}{\epsilon}\left(f\left(\frac{x}{\epsilon}, u_0\right) - f_0(u_0)\right) + e_\epsilon \text{ on } \Gamma, \qquad (3.1)\\ &-\frac{\partial u_\epsilon^{(1)}}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma, \end{aligned}$$

$$\int_\Gamma u_\epsilon^{(1)}\, d\sigma_x = 0, \qquad (3.2)$$

where

$$e_\epsilon = \frac{1}{\epsilon}\int_\Gamma \left(f_0(u_0) - f\left(\frac{x}{\epsilon}, u_0\right)\right) d\sigma_x.$$

Hence e_ϵ is chosen such that the solution always exists, and the condition (3.2) guarantees this solution is unique. We note that given u_0, this is a *linear* problem. Now if u_ϵ and u_0 are in $L^\infty(\Gamma)$, let

$$D_\epsilon = \max\left\{\|u_\epsilon\|_{L^\infty(\Gamma)}, \|u_0\|_{L^\infty(\Gamma)}\right\} \tag{3.3}$$

and let

$$M_\epsilon = \sup_{(y,w)\in Y\times[-D_\epsilon,D_\epsilon]} \frac{\partial f}{\partial v}(y,w). \tag{3.4}$$

The next estimate holds for dimension $n = 2$ or 3 but depends on the constant M_ϵ. We do *not* know a priori that D_ϵ is finite in general when $n = 3$. However, such an assumption seems to be physically reasonable and known to be the case when the medium is layered.

PROPOSITION 3.1. *Let $n = 2$ or 3 and let u_ϵ, u_0 be minimizers of* (2.1), (2.7), *respectively, and let $u_\epsilon^{(1)}$ be the solution to* (3.1). *Assume also that $u_\epsilon \in C^0(\bar{\Omega})$. Then there exists constants C and D independent of ϵ such that*

$$\|u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)}\|_{H^1(\Omega)} \le C\epsilon(M_\epsilon + D),$$

where M_ϵ is defined by (3.4). *Furthermore, there exist constants* C_1 *and* C_2 *independent of ϵ such that*

$$\|u_\epsilon^{(1)}\|_{L^2(\Gamma)} \le \mathrm{C}_1 \quad \textit{and} \quad |e_\epsilon| \le \mathrm{C}_2.$$

Proof. Let

$$z_\epsilon = u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)}.$$

Since u_ϵ is continuous, by (2.6), we have that for any $v \in H^1(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla z_\epsilon \cdot \nabla v \; dx &= \int_\Omega \nabla u_\epsilon \cdot \nabla v \; dx - \int_\Omega \nabla u_0 \cdot \nabla v \; dx - \epsilon \int_\Omega \nabla u_\epsilon^{(1)} \cdot \nabla v \; dx \\ &= -\int_\Gamma f\left(\frac{x}{\epsilon}, u_\epsilon\right) v d\sigma_x + \int_\Gamma f\left(\frac{x}{\epsilon}, u_0\right) v d\sigma_x + \epsilon \int_\Gamma e_\epsilon v d\sigma_x.\end{aligned}$$

So,

$$\int_\Omega \nabla z_\epsilon \cdot \nabla v \; dx + \int_\Gamma \left[f\left(\frac{x}{\epsilon}, u_\epsilon\right) - f\left(\frac{x}{\epsilon}, u_0\right)\right] v d\sigma_x - \epsilon \int_\Gamma e_\epsilon v d\sigma_x = 0.$$

Now note that u_0 and u_ϵ are defined pointwise on Γ. So, by the mean value theorem, for each fixed ϵ and $x \in \Gamma$ there exists ξ_ϵ^x between $u_0(x)$ and $u_\epsilon(x)$ such that

$$f\left(\frac{x}{\epsilon}, u_\epsilon\right) - f\left(\frac{x}{\epsilon}, u_0\right) = (u_\epsilon - u_0)\frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right).$$

By subtracting and adding $\epsilon u_\epsilon^{(1)}$ we have

$$f\left(\frac{x}{\epsilon}, u_\epsilon\right) - f\left(\frac{x}{\epsilon}, u_0\right) = z_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) + \epsilon u_\epsilon^{(1)} \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right),$$

which, if we pick $v = z_\epsilon$, yields

$$\int_\Omega |\nabla z_\epsilon|^2 \; dx + \int_\Gamma z_\epsilon^2 \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) d\sigma_x = -\epsilon \int_\Gamma u_\epsilon^{(1)} \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) z_\epsilon d\sigma_x + \epsilon e_\epsilon \int_\Gamma z_\epsilon d\sigma_x.$$

Since $\frac{\partial f}{\partial v} \geq c_0$, this implies

$$\tilde{c_0}\|z_\epsilon\|^2_{H^1(\Omega)} \leq \int_\Omega |\nabla z_\epsilon|^2 \ dx + \int_\Gamma z_\epsilon^2 \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) d\sigma_x$$
$$= -\epsilon \int_\Gamma u_\epsilon^{(1)} \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) z_\epsilon d\sigma_x + \epsilon e_\epsilon \int_\Gamma z_\epsilon d\sigma_x.$$

So by applying Hölder's inequality and then the trace theorem we have

$$\tilde{c_0}\|z_\epsilon\|^2_{H^1(\Omega)} \leq \epsilon \left\| \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right)\right\|_{L^\infty(\Gamma)} \|u_\epsilon^{(1)}\|_{L^2(\Gamma)} \|z_\epsilon\|_{L^2(\Gamma)} + \epsilon |e_\epsilon| |\Gamma|^{1/2} \|z_\epsilon\|_{L^2(\Gamma)}$$
$$\leq \epsilon \left(\left\| \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right)\right\|_{L^\infty(\Gamma)} \|u_\epsilon^{(1)}\|_{L^2(\Gamma)} + |e_\epsilon||\Gamma|^{1/2}\right) \|z_\epsilon\|_{H^1(\Omega)}.$$

Thus, we can write

$$\|z_\epsilon\|_{H^1(\Omega)} \leq C\epsilon \left(\left\| \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right)\right\|_{L^\infty(\Gamma)} \|u_\epsilon^{(1)}\|_{L^2(\Gamma)} + |e_\epsilon| \right). \tag{3.5}$$

Now recall for any v we have

$$\int_Y (f(y,v) - f_0(v)) dy = 0,$$

so there exists a continuous Y-periodic function $r(y,v)$ such that

$$\Delta_y r(y,v) = f(y,v) - f_0(v) \quad \forall v \in R. \tag{3.6}$$

So we have

$$e_\epsilon = \frac{1}{\epsilon} \int_\Gamma \left(f_0(u_0) - f\left(\frac{x}{\epsilon}, u_0\right)\right) d\sigma_x$$
$$= -\frac{1}{\epsilon}\int_\Gamma \Delta_y r\left(\frac{x}{\epsilon}, u_0\right) d\sigma_x$$
$$= -\int_{\partial\Gamma} \nabla_y r\left(\frac{x}{\epsilon}, u_0\right) \cdot \nu \ ds_x,$$

where the last equality is arrived at using integration by parts and the fact that the chain rule implies $\frac{\partial r}{\partial y}(x/\epsilon, u_0) = \epsilon \frac{\partial r}{\partial x}(x/\epsilon, u_0)$. Note that the differential operators ∇_y and Δ_y are with respect to $y \in Y$; that is, they are surface operators. Now since u_0 is bounded pointwise on Γ and since $r(y,v)$ is a continuously differentiable Y-periodic function we have

$$e_\epsilon \leq C, \tag{3.7}$$

where C is bounded independent of ϵ. Now we show that $\|u_\epsilon^{(1)}\|_{L^2(\Gamma)}$ is similarly bounded. Let $w_\epsilon \in H^1(\Omega)$ satisfy

$$\begin{aligned} \Delta w_\epsilon &= 0 \text{ in } \Omega, \\ \frac{\partial w_\epsilon}{\partial n} &= u_\epsilon^{(1)} \text{ on } \Gamma, \\ \frac{\partial w_\epsilon}{\partial n} &= 0 \text{ on } \partial\Omega \setminus \Gamma, \\ \int_\Gamma w_\epsilon \ d\sigma_x &= 0; \end{aligned} \tag{3.8}$$

then

$$\int_\Gamma (u_\epsilon^{(1)})^2 d\sigma_x = \int_\Gamma u_\epsilon^{(1)} \frac{\partial w_\epsilon}{\partial n} d\sigma_x = \int_\Omega \nabla u_\epsilon^{(1)} \nabla w_\epsilon \, dx = \int_{\partial\Omega} \frac{\partial u_\epsilon^{(1)}}{\partial n} w_\epsilon d\sigma_x,$$

where the last two equalities follow from integration by parts. Now, since $u_\epsilon^{(1)}$ satisfies (3.1), we have

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u_\epsilon^{(1)}}{\partial n} w_\epsilon d\sigma_x &= -\int_\Gamma \left[\frac{f(x/\epsilon, u_0) - f_0(u_0)}{\epsilon} + e_\epsilon\right] w_\epsilon d\sigma_x \\ &= -\frac{1}{\epsilon}\int_\Gamma \Delta_y r\left(\frac{x}{\epsilon}, u_0\right) w_\epsilon d\sigma_x - e_\epsilon \int_\Gamma w_\epsilon d\sigma_x \\ &= -\frac{1}{\epsilon}\int_\Gamma \Delta_y r\left(\frac{x}{\epsilon}, u_0\right) w_\epsilon d\sigma_x,\end{aligned}$$

where the second equality follows from (3.6) and the last equality holds since $\int_\Gamma w_\epsilon d\sigma_x = 0$. Now using the chain rule we can write

$$\Delta_y r(x/\epsilon, u_0) = \epsilon^2 \Delta_x r(x/\epsilon, u_0),$$

where Δ_x is a surface Laplacian on Γ. Thus, we have

$$\begin{aligned}\int_\Gamma (u_\epsilon^{(1)})^2 d\sigma_x &= -\epsilon \int_\Gamma \Delta_x r\left(\frac{x}{\epsilon}, u_0\right) w_\epsilon d\sigma_x \\ &= \epsilon \int_\Gamma \nabla_x r\left(\frac{x}{\epsilon}, u_0\right) \nabla w_\epsilon d\sigma_x - \epsilon \int_{\partial\Gamma} \frac{\partial_x r}{\partial \nu} w_\epsilon ds_x,\end{aligned} \tag{3.9}$$

where ν is the outward unit normal to $\partial\Gamma$. Note that when $n = 2$, we use the last integral to represent endpoint evaluation. So, by Hölder's inequality,

$$\begin{aligned}\epsilon \int_\Gamma \nabla_x r\left(\frac{x}{\epsilon}, u_0\right) \nabla w_\epsilon d\sigma_x - \epsilon \int_{\partial\Gamma} \frac{\partial_x r}{\partial \nu} w_\epsilon ds_x \; &\le \epsilon \|\nabla_x r\|_{L^2(\Gamma)} \|\nabla w_\epsilon\|_{L^2(\Gamma)} \\ &+ \epsilon \left\| \frac{\partial_x r}{\partial \nu} \right\|_{L^2(\partial\Gamma)} \|w_\epsilon\|_{L^2(\partial\Gamma)}.\end{aligned} \tag{3.10}$$

Then by the trace theorem we have

$$\|w_\epsilon\|_{L^2(\partial\Gamma)} \le \|w_\epsilon\|_{H^1(\Gamma)} \le \|w_\epsilon\|_{H^{3/2}(\Omega)}. \tag{3.11}$$

Similarly,

$$\|\nabla w_\epsilon\|_{L^2(\Gamma)} \le \|w_\epsilon\|_{H^1(\Gamma)} \le \|w_\epsilon\|_{H^{3/2}(\Omega)}. \tag{3.12}$$

Then (3.9), (3.10), (3.11), and (3.12) imply

$$\|u_\epsilon^{(1)}\|^2_{L^2(\Gamma)} \le \epsilon \left(\|\nabla_x r\|_{L^2(\Gamma)} + \left\| \frac{\partial_x r}{\partial \nu} \right\|_{L^2(\partial\Gamma)} \right) \|w_\epsilon\|_{H^{3/2}(\Omega)}.$$

Now since w_ϵ satisfies (3.8) we have from standard elliptic regularity theory [7]

$$\|w_\epsilon\|_{H^{3/2}(\Omega)} \le C \|u_\epsilon^{(1)}\|_{L^2(\Gamma)},$$

where C is independent of ϵ and so we can write

$$\|u_\epsilon^{(1)}\|_{L^2(\Gamma)} \leq C\epsilon \left(\left\| \nabla_x r\left(\frac{x}{\epsilon}, u_0\right) \right\|_{L^2(\Gamma)} + \left\| \frac{\partial_x r(x/\epsilon, u_0)}{\partial \nu} \right\|_{L^2(\partial\Gamma)} \right)$$
$$= C \left(\left\| \nabla_y r\left(\frac{x}{\epsilon}, u_0\right) \right\|_{L^2(\Gamma)} + \left\| \frac{\partial_y r(x/\epsilon, u_0)}{\partial \nu} \right\|_{L^2(\partial\Gamma)} \right),$$

where the last equality follows from the chain rule. Consequently, since we have that u_0 is continuous on Γ and bounded pointwise and since $r(y, v)$ is a continuously differentiable Y-periodic function we can conclude that

$$\|u_\epsilon^{(1)}\|_{L^2(\Gamma)} \leq D, \tag{3.13}$$

where D is bounded independently of ϵ. Then (3.5), (3.7), and (3.13) imply the main result of the proposition:

$$\|z_\epsilon\|_{H^1(\Omega)} \leq C\epsilon \left(\left\| \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \xi_\epsilon^x\right) \right\|_{L^\infty(\Gamma)} \|u_\epsilon^{(1)}\|_{L^2(\Gamma)} + |e_\epsilon| \right) \leq \tilde{C}\epsilon(M_\epsilon + \tilde{D}),$$

where M_ϵ is defined by (3.4). □

Note that in light of Lemma 2.3, we can easily establish the following corollaries.

COROLLARY 3.2. *When $n = 2$, i.e., for the case in which $\Omega \subset R^2, \Gamma \subset R$ with boundary period cell $Y = [0, 1]$, there exists a constant C independent of ϵ such that*

$$\|u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)}\|_{H^1(\Omega)} \leq C\epsilon. \tag{3.14}$$

COROLLARY 3.3. *When $n = 2$, for u_ϵ the weak solution of* (1.5) *and u_0 the weak solution of* (1.6), *there exists a constant C independent of ϵ such that*

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \leq C\sqrt{\epsilon}. \tag{3.15}$$

Estimate (3.15) follows from the fact that

$$\|u_\epsilon^{(1)}\|_{H^1(\Omega)} \leq C \left\| \frac{\partial u_\epsilon^{(1)}}{\partial n} \right\|_{H^{-1/2}(\Gamma)} \leq C\epsilon^{-1/2},$$

where the last inequality follows by interpolating between $L^2(\Gamma)$ and $H^1(\Gamma)$ (see [8, section 11.5]) and then using duality (as in [11]). Finally, note that estimate (3.14) also holds for $n = 3$ if we know that D_ϵ defined by (3.3) is uniformly bounded.

4. Numerical experiments. Here we will both test the accuracy of our asymptotic expansion and observe the behavior of the current by performing numerical experiments in two dimensions. Note that for the two dimensional problem the domain Ω is a unit square and the boundary Γ is the right side of the unit square, that is,

$$\Gamma = \{(x_1, x_2) : x_1 = 1\}$$

(see Figure 2.1). To compute solutions u_ϵ, u_0, and $u_\epsilon^{(1)}$, we use piecewise linear finite elements on a regular mesh. To avoid singularities within elements, we choose a grid which conforms to the medium. To perform the nonlinear minimization (when solving for u_ϵ), we use a conjugate gradient descent based algorithm developed by Hager and

TABLE 4.1
Table of estimates over Ω and convergence rates.

ϵ	1/5	1/11	1/25	1/40	α		
$\|u_\epsilon - (u_0 + \epsilon u_\epsilon^{(1)})\|_{H^1(\Omega)}$	.0189	.0090	.0040	.0025	.9699	.9843	.9913
$\|u_\epsilon - u_0\|_{H^1(\Omega)}$	.0537	.0360	.0238	.0188	.5057	.5061	.506 0
$\|u_\epsilon - u_0\|_{L^2(\Omega)}$	.0063	.0027	.0011	.0007	1.0808	1.0722	1.0676

TABLE 4.2
Table of estimates over Γ and estimates of the gradient over Γ.

ϵ	1/5	1/11	1/25	1/40
$\|u_\epsilon - (u_0 + \epsilon u_\epsilon^{(1)})\|_{L^2(\Gamma)}$	0.0108	0.0050	0.0022	0.0014
$\|u_\epsilon - u_0\|_{L^2(\Gamma)}$	0.0128	0.0057	0.0025	0.0015
$\|\nabla u_\epsilon - \nabla(u_0 + \epsilon u_\epsilon^{(1)})\|_{L^2(\Gamma)}$	0.1027	0.0710	0.0475	0.0377
$\|\nabla u_\epsilon - \nabla u_0\|_{L^2(\Gamma)}$	0.1235	0.0817	0.0536	0.0422

Zhang [5]. Note that the homogenized solution u_0 is simply a constant value here, which we can find by Newton's method. The correction, $u_\epsilon^{(1)}$, is computed using standard finite elements for a linear problem, again conforming to the media.

We perform these computations for $\epsilon = 1/5$, $\epsilon = 1/11$, $\epsilon = 1/25$, and $\epsilon = 1/40$. We use the following parameter values for our simulation: $J_A = 1$, $J_C = 10$, $V_A = 0.5$, $V_C = 1.0$, $\alpha_{aa} = 0.5$, $\alpha_{ca} = 0.85$, and $Y = Y_A \bigcup Y_C$, where $Y_A = [0, 1/3]$ and $Y_C = [1/3, 1]$. Note that for the parameter values used in this implementation, we have $u_0 = 0.9758$. We have shown analytically that the estimates below hold for the case of layered media and wish to numerically verify these estimates:

$$\|u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)}\|_{H^1(\Omega)} \le \mathrm{C}_1\epsilon,$$
$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \le \mathrm{C}_2\sqrt{\epsilon}.$$

The results are summarized in Table 4.1. The estimates above are all bounded by a term of the form $C\epsilon^\alpha$. We estimate this exponent α in Table 4.1. Note that the numerical results in Table 4.1 are in compliance with the given estimates.

In Figures 4.1 and 4.2 we plot the "correct" and asymptotic approximation of the potential on Ω when $\epsilon = 1/5$. We see that the macroscopic behavior is captured by the expansion. Figures 4.3 and 4.4 show the same for $\epsilon = 1/11$. In Figure 4.5(a)–4.5(d) we can view the limiting behavior of u_ϵ on Γ as ϵ approaches 0. To examine the influence of the corrector term more closely, in Figures 4.6–4.9 we graph both the "correct" solution and the asymptotic expansion over Γ with material regions indicated. Note that the asymptotic approximation is not exact and in fact is slightly skewed. This is probably due to the linearization of the corrector term. In Figure 4.10 we graph the L^∞-norm of ∇u_ϵ on the boundary for various values of ϵ. We see that according to our simulations of the layered media case, the current remains bounded as the perimeter becomes arbitrarily large, suggesting that the linear relation between current and perimeter observed in [9] may not hold for all geometries. Our results, however, do not directly contradict the observations made in [9], where the computations were done for a fixed number of anodes with a varying geometry. Furthermore, since the estimates here are merely in $H^1(\Omega)$, pointwise estimates for the gradient (current) on the boundary do not follow.

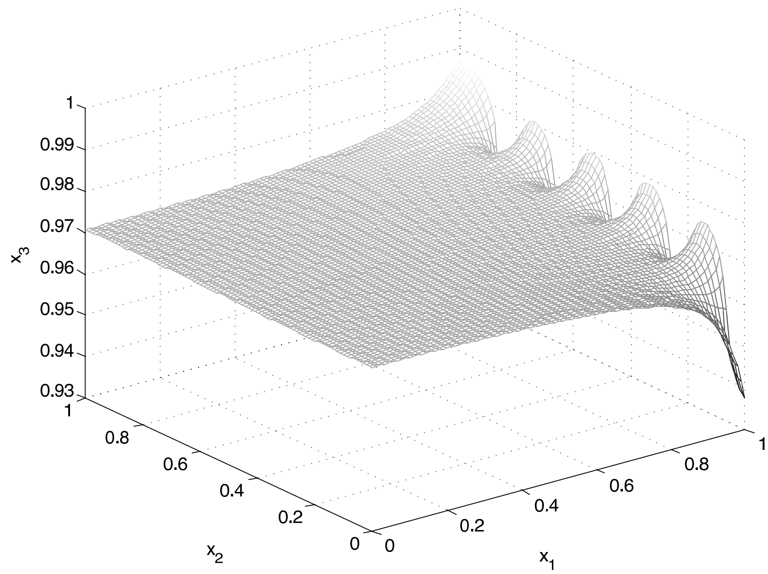

FIG. 4.1. u_ϵ, $\epsilon = 1/5$.

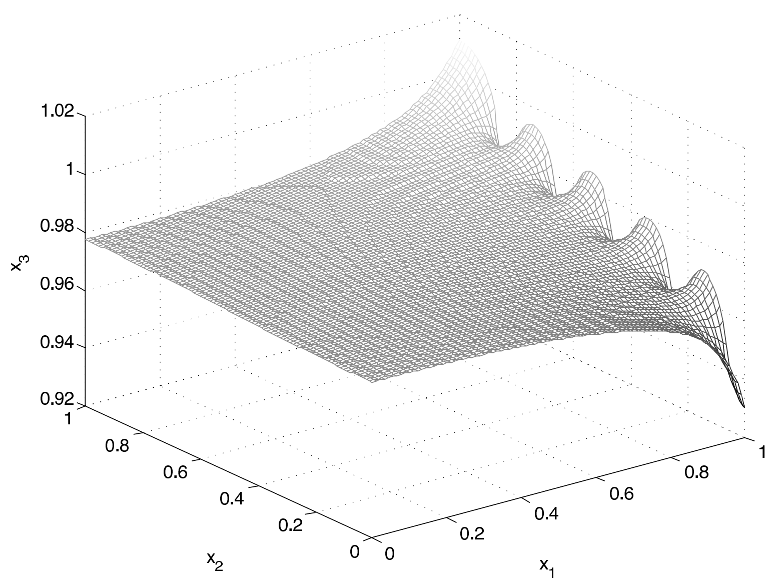

FIG. 4.2. $u_0 + \epsilon u_\epsilon^{(1)}$, $\epsilon = 1/5$.

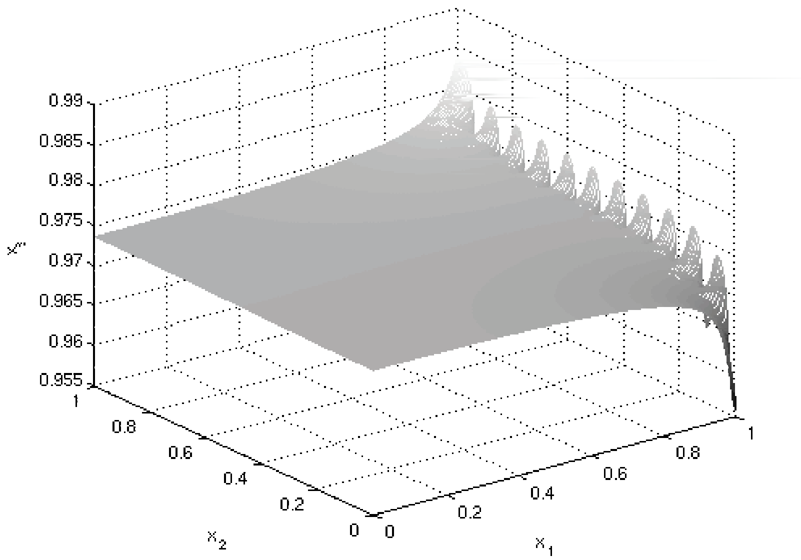

FIG. 4.3. u_ϵ, $\epsilon = 1/11$.

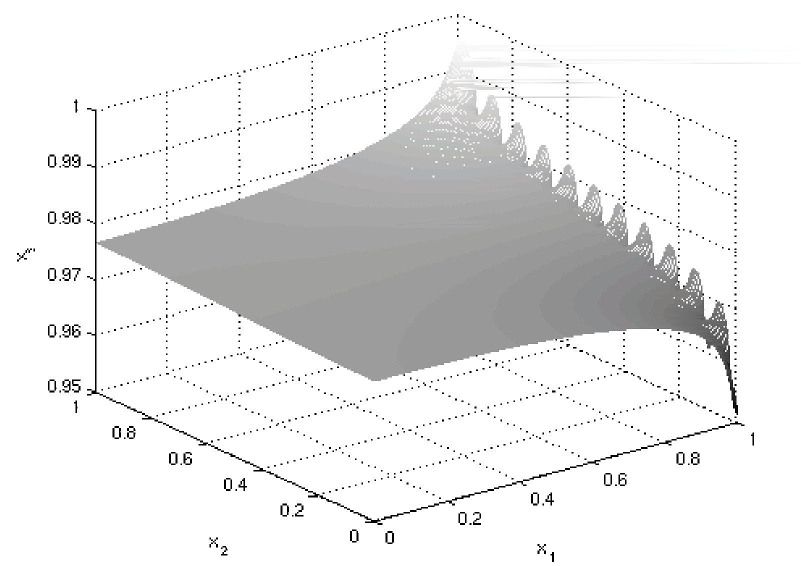

FIG. 4.4. $u_0 + \epsilon u_\epsilon^{(1)}$, $\epsilon = 1/11$.

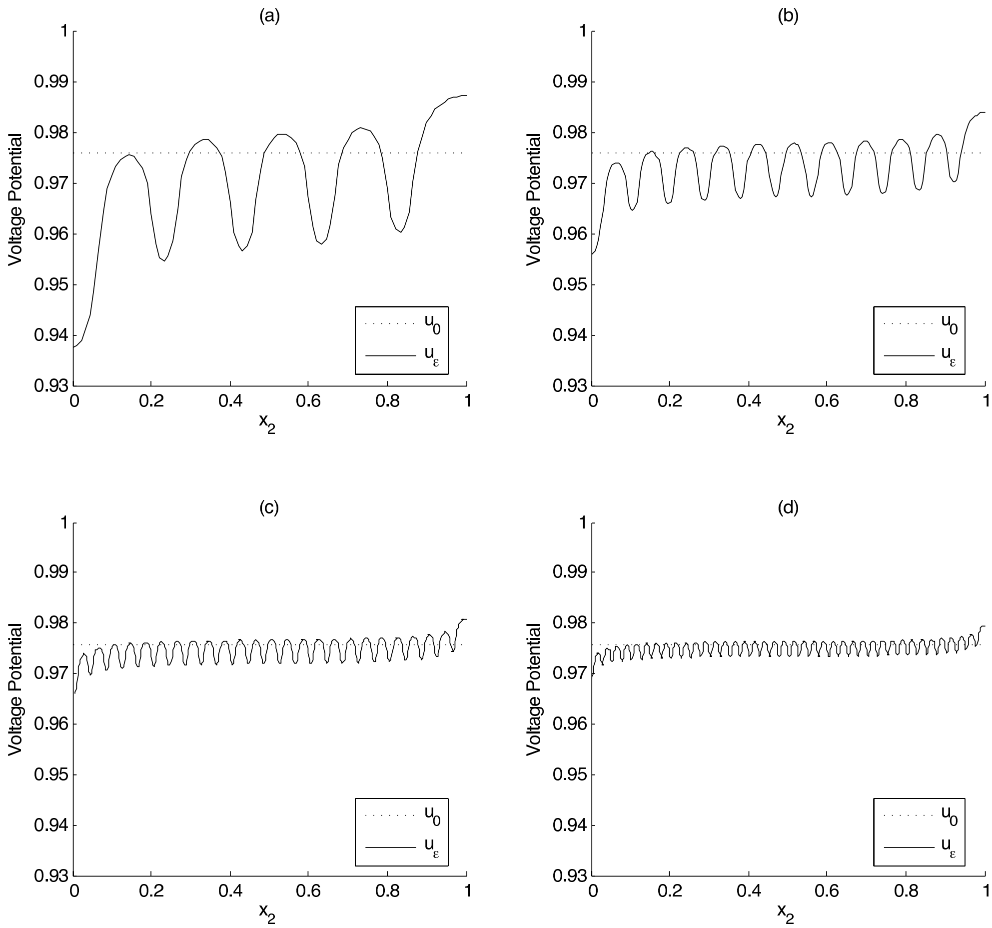

FIG. 4.5. *Limiting behavior of u_ϵ on Γ as ϵ approaches zero for* (a) $\epsilon = 1/5$, (b) $\epsilon = 1/11$, (c) $\epsilon = 1/25$, (d) $\epsilon = 1/40$.

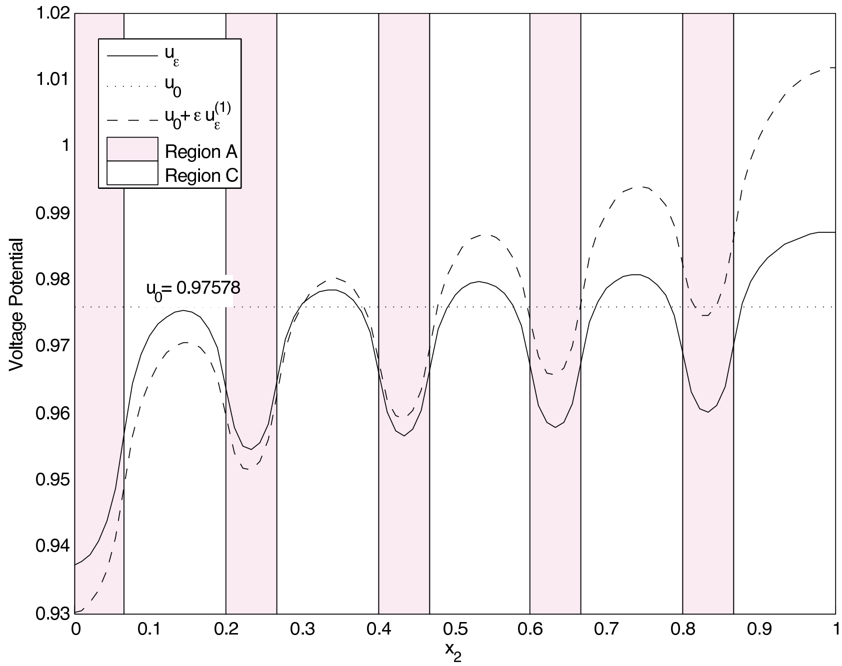

FIG. 4.6. *The potential on the boundary* Γ, $\epsilon = 1/5$.

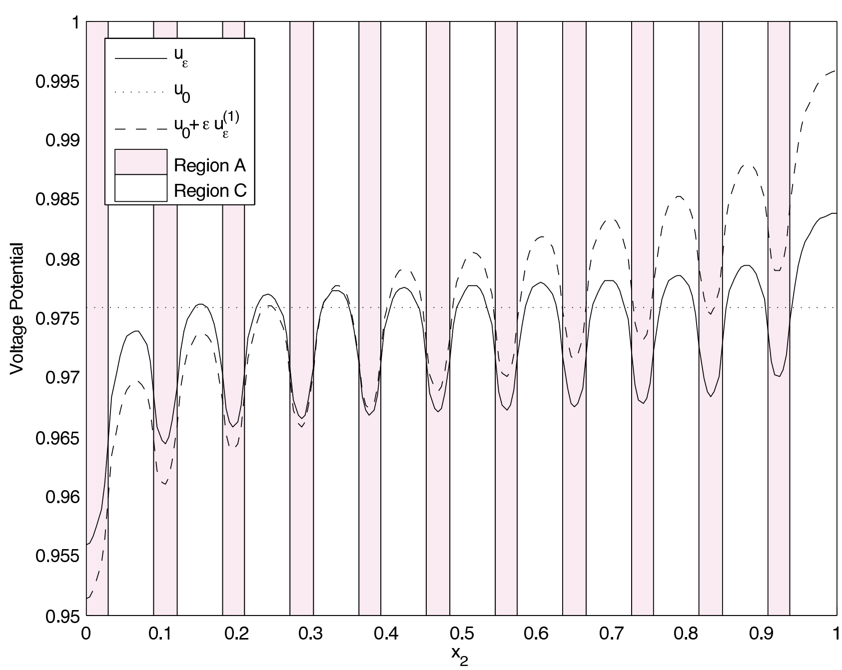

FIG. 4.7. *The potential on the boundary* Γ, $\epsilon = 1/11$.

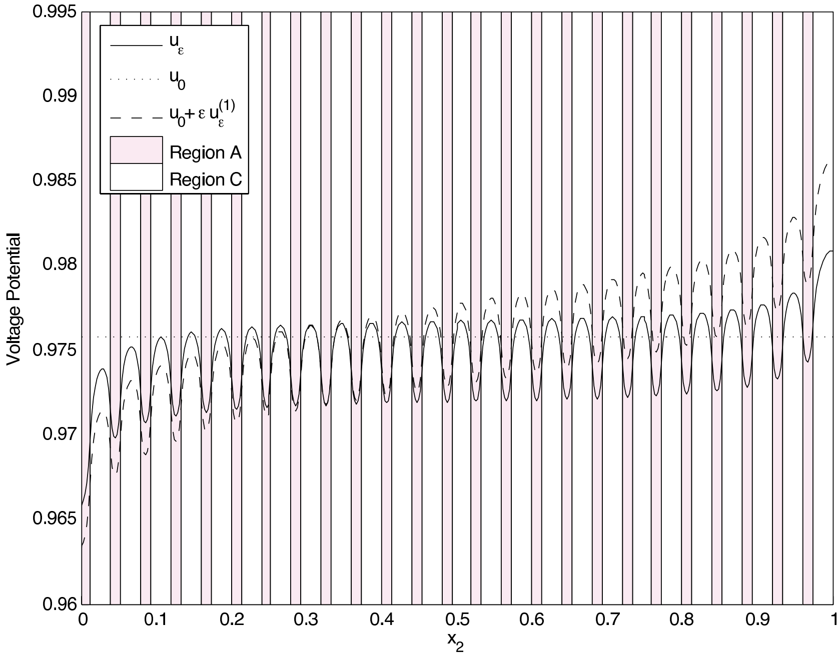

FIG. 4.8. *The potential on the boundary* Γ, $\epsilon = 1/25$.

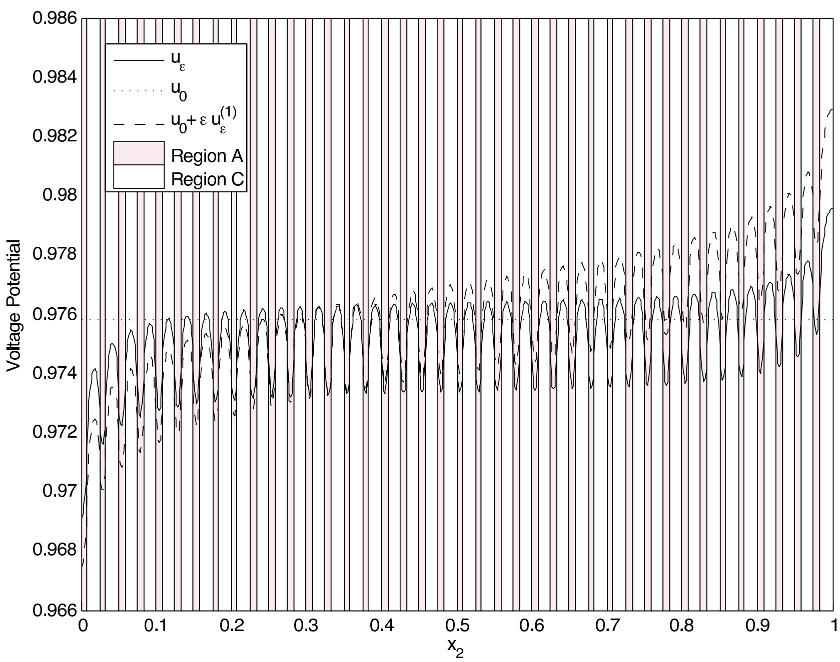

FIG. 4.9. *The potential on the boundary* Γ, $\epsilon = 1/40$.

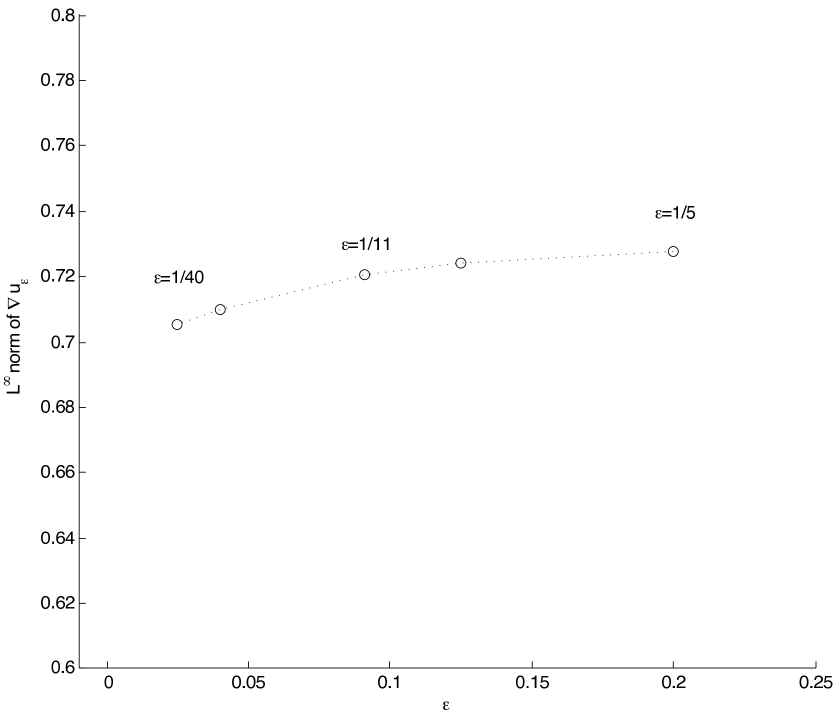

FIG. 4.10. *L^∞-norm of ∇u_ϵ on Γ as ϵ approaches* 0.

5. Conclusion. We have analyzed a Butler–Volmer-type model which describes the potential distribution in a system of anodic islands in a coplanar cathodic matrix with a periodic structure. By using a multiscale approach we have determined the limiting problem for the boundary value problem (1.5) as the period approaches zero. Furthermore, by introducing a linear correction, we have developed an asymptotic expansion which closely estimates the solution of the original boundary value problem. Essentially, we have taken a *nonlinear heterogeneous* problem and decomposed it, in a sense, into a *nonlinear homogeneous* problem and a *linear heterogeneous* problem. Hence the homogenization approach to this problem gives insight into the behavior of the solution while also providing an efficient computational technique. The corrector term, although inhomogeneous, solves a linear problem and was therefore not difficult to compute in our experiments. However, in higher dimensions or for very small scale problems, one may want to homogenize the corrector term itself. This could perhaps be done by solving a cell problem or looking at the tail behavior, as in [1] or [2]. In this paper we have used the language and terminology of galvanic corrosion, but this analysis could also carry over to a more general class of elliptic problems with nonlinear boundary conditions having periodic structure (assuming the appropriate convexity conditions). Future work must address the continuity and boundedness issues of the three dimensional problem; i.e., the lack of an applicable Orlicz estimate must be resolved. Also of interest in future work would be the development of a better corrector term, thereby improving the accuracy of the asymptotic approximation.

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